Problem 8

(a) Prove a formula similar to the one in Problem 7(a) but involving arccot instead of arctan.

(b) Find the sum of the series $\sum_{n=0}^{\infty} \operatorname{arccot}(n^2+n+1)$.

Solution

Part (a)

Consider the difference formula for the tangent function.

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

Since

$$\cot x = \frac{1}{\tan x},$$

we can derive the corresponding difference formula for cotangent.

$$\begin{aligned}
\cot(\alpha-\beta) &= \frac{1}{\tan(\alpha-\beta)} \\
&= \frac{1+\tan\alpha\tan\beta}{\tan\alpha-\tan\beta} \\
&= \frac{1+\tan\alpha\tan\beta}{\tan\alpha-\tan\beta}\cdot\frac{\frac{1}{\tan\alpha\tan\beta}}{\frac{1}{\tan\alpha\tan\beta}} \\
&= \frac{\frac{1}{\tan\alpha\tan\beta}+1}{\frac{1}{\tan\beta}-\frac{1}{\tan\alpha}} \\
&= \frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}
\end{aligned}$$

The corresponding difference formula for cotangent is thus

$$\cot(\alpha-\beta) = \frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}.$$

Now armed with it, we are in a position to find a similar formula to the one in Problem 7(a). It will look something like

$$\operatorname{arccot} x - \operatorname{arccot} y = \operatorname{arccot}\frac{f(x,y)}{g(x,y)}.$$

Hence, consider the cotangent of the left-hand side.

$$\begin{aligned}
\cot(\operatorname{arccot} x - \operatorname{arccot} y) &= \frac{\cot(\operatorname{arccot} x)\cot(\operatorname{arccot} y)+1}{\cot(\operatorname{arccot} y)-\cot(\operatorname{arccot} x)} \\
&= \frac{xy+1}{y-x}
\end{aligned}$$

Therefore,

$$\cot(\operatorname{arccot} x - \operatorname{arccot} y) = \frac{xy+1}{y-x}.$$

Take the arccotangent of both sides to get the desired result.

$$\operatorname{arccot} x - \operatorname{arccot} y = \operatorname{arccot} \frac{xy+1}{y-x},$$

where $y \neq x$.

Part (b)

The series whose sum we have to find is the following.

$$\sum_{n=0}^{\infty} \operatorname{arccot}(n^2+n+1)$$

Since we just have one arccot term in the summand, we'll use the formula we derived in part (a) to break it into two terms. The argument of arccot is actually a fraction.

$$\sum_{n=0}^{\infty} \operatorname{arccot}\left(\frac{n^2+n+1}{1}\right)$$

Thus, we set

$$\begin{aligned} xy+1 &= n^2+n+1 \\ y-x &= 1 \end{aligned}$$

and solve for x and y with substitution. Doing this gives $x = n$ and $y = n+1$. By the formula in part (a), then, we have

$$\operatorname{arccot} \frac{n^2+n+1}{1} = \operatorname{arccot} n - \operatorname{arccot}(n+1),$$

so the series we have to evaluate becomes

$$\sum_{n=0}^{\infty} [\operatorname{arccot} n - \operatorname{arccot}(n+1)].$$

Write out the first three terms of it.

$$\sum_{n=0}^{\infty} [\operatorname{arccot} n - \operatorname{arccot}(n+1)] = \underbrace{\operatorname{arccot} 0 - \cancel{\operatorname{arccot} 1}}_{n=0} + \underbrace{\cancel{\operatorname{arccot} 1} - \cancel{\operatorname{arccot} 2}}_{n=1} + \underbrace{\cancel{\operatorname{arccot} 2} - \operatorname{arccot} 3}_{n=2} + \cdots$$

Every value of n gives us two terms. The second term of n always cancels with the first term of $n+1$. Hence, this is a telescoping series, which we evaluate by calculating a limit.

$$\begin{aligned} \sum_{n=0}^{\infty} [\operatorname{arccot} n - \operatorname{arccot}(n+1)] &= \operatorname{arccot} 0 - \lim_{n\to\infty} \operatorname{arccot}(n+1) \\ &= \frac{\pi}{2} - 0 \\ &= \frac{\pi}{2} \end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty} \operatorname{arccot}(n^2+n+1) = \frac{\pi}{2}.$$